

$$\ddot{\theta} + \frac{g}{l} \sin \theta = \frac{1}{ml^2} \tau$$

$$\theta = \theta(t)$$

$$\tau = \tau(t)$$

$$\dot{\theta} = \ddot{\theta} = 0$$

Static Equilibrium,

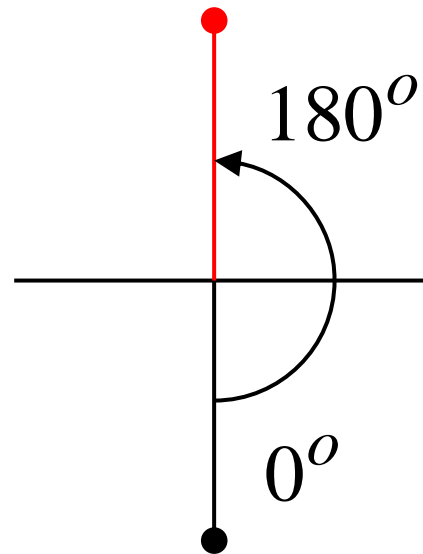
$$\theta_0, \tau_0$$

$$\sin \theta_0 = \frac{l}{gml^2} \tau_0 = \frac{1}{gml} \tau_0$$

Assume  $\tau_0 = 0$

$$\Rightarrow \sin \theta_0 = 0$$

$$\Rightarrow \theta_0 = 0^\circ \text{ or } 180^\circ$$



# LINEARIZATION → Taylor's Series Expansion

$$\ddot{\theta} + \frac{g}{l} \sin \theta - \frac{1}{ml^2} \tau = 0 = f(\ddot{\theta}, g, l, m, \tau, \theta)$$

$$f - f_0 = \left. \frac{\partial f}{\partial \ddot{\theta}} \right|_0 (\ddot{\theta} - \ddot{\theta}_0) + \left. \frac{\partial f}{\partial g} \right|_0 \underbrace{(g - g_0)}_{\Delta g = 0} + \left. \frac{\partial f}{\partial l} \right|_0 \underbrace{(l - l_0)}_{\Delta l = 0}$$

$$+ \left. \frac{\partial f}{\partial m} \right|_0 \underbrace{(m - m_0)}_{\Delta m = 0} + \left. \frac{\partial f}{\partial \tau} \right|_0 (\tau - \tau_0) + \left. \frac{\partial f}{\partial \theta} \right|_0 (\theta - \theta_0)$$

$$\Delta f = 0 = 1 \Delta \ddot{\theta} + \left( -\frac{1}{ml^2} \right)_0 \Delta \tau + \left( \frac{g}{l} \cos \theta \right)_0 \Delta \theta$$

$$\Delta \ddot{\theta} + \left( \frac{g}{l} \cos \theta \right)_0 \Delta \theta = \left( \frac{1}{ml^2} \right)_0 \Delta \tau$$

If the equilibrium point is  $\theta = 0^\circ \Rightarrow$

$$\Delta \ddot{\theta} + \left( \frac{g}{l} \cos 0 \right) \Delta \theta = \frac{1}{ml^2} \Delta \tau$$

$$\Delta \ddot{\theta} + \frac{g}{l} \Delta \theta = \frac{1}{ml^2} \Delta \tau$$

If the equilibrium point is  $\theta = 180^\circ \Rightarrow$

$$\Delta \ddot{\theta} + \left( \frac{g}{l} \cos 180 \right) \Delta \theta = \frac{1}{ml^2} \Delta \tau$$

$$\Delta \ddot{\theta} - \frac{g}{l} \Delta \theta = \frac{1}{ml^2} \Delta \tau$$

## LAPLACE TRANSFORM (Assume Zero ICs)

$$\Delta \ddot{\theta} + \frac{g}{l} \Delta \theta = \frac{1}{ml^2} \Delta \tau \quad \Rightarrow \text{Laplace Transform}$$

$$\Delta \Theta(s)s^2 + \frac{g}{l} \Delta \Theta(s) = \frac{1}{ml^2} \Delta \Gamma$$

$$\left( s^2 + \frac{g}{l} \right) \Delta \Theta = \frac{1}{ml^2} \Delta \Gamma$$

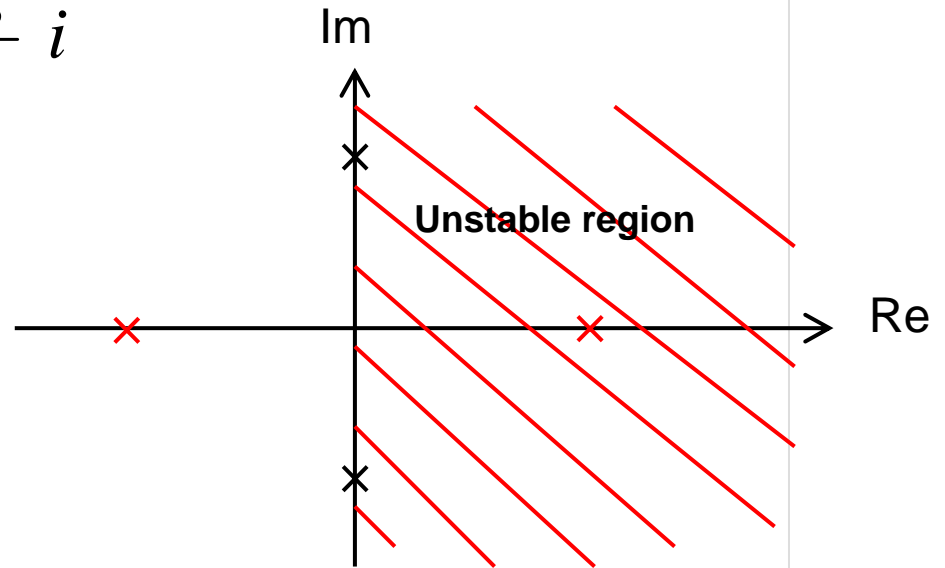

$$\frac{\Delta \Theta}{\Delta \Gamma} = \frac{1}{ml^2} \left( \frac{1}{s^2 + \frac{g}{l}} \right) \quad \text{or} \quad \Delta \Theta = \frac{1}{ml^2} \left( \frac{1}{s^2 + \frac{g}{l}} \right) \Delta \Gamma$$

$$s^2 + \frac{g}{l} = 0 \Rightarrow \text{characteristic equation}$$

$$\text{eigen values} \Rightarrow s = \pm \sqrt{\frac{g}{l}} i$$

$$\text{C.E. for } \theta_0 = 180^\circ \Rightarrow$$

$$s^2 - \frac{g}{l} = 0 \Rightarrow s = \pm \sqrt{\frac{g}{l}}$$



If the eigen values have

- positive real parts, then the system is unstable
- negative real parts, then the system is stable
- zero real part, then the system is marginally stable



$$\Delta \ddot{\theta} + \frac{g}{l} \Delta \theta = \frac{1}{ml^2} \Delta \tau \quad \Rightarrow \quad \text{State space representation}$$

Set of first order DFQs

$$z_1 = \Delta \theta$$

$$z_2 = \dot{z}_1 = \Delta \dot{\theta} \quad \Rightarrow \quad \boxed{\dot{z}_1 = z_2}$$

Number of DFQs = Order of system

$$\dot{z}_2 + \frac{g}{l} z_1 = \frac{1}{ml^2} \Delta \tau$$

$$\boxed{\dot{z}_2 = -\frac{g}{l} z_1 + \frac{1}{ml^2} \Delta \tau}$$

$$\dot{z}_1 = 0z_1 + 1z_2 + 0\Delta\tau$$

$$\dot{z}_2 = -\frac{g}{l}z_1 + 0z_2 + \frac{1}{ml^2}\Delta\tau$$

$\dot{x} = Ax + Bu$  where  $A =$  coefficient matrix – square

$B =$  order of the system

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l} & 0 \end{bmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{ml^2} \end{pmatrix} \Delta\tau$$

Output :  $y = Cx + Du$

$$\text{C.E.} \Rightarrow \det(SI - A) = 0$$

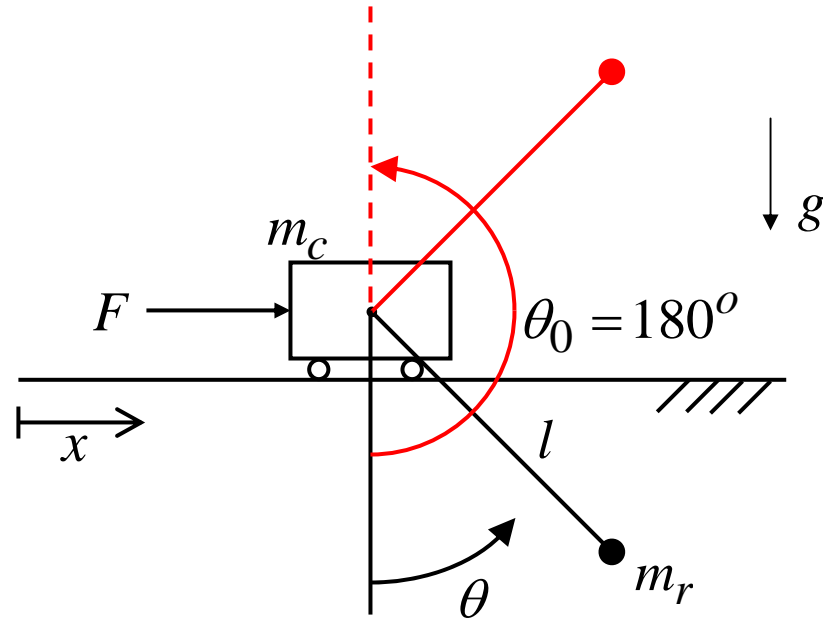
$$\det \begin{bmatrix} \left| \begin{array}{cc} s & 0 \\ 0 & s \end{array} \right| - \left| \begin{array}{cc} 0 & 1 \\ -\frac{g}{l} & 0 \end{array} \right| \end{bmatrix} = 0$$

$$\det \begin{vmatrix} s & -1 \\ \frac{g}{l} & s \end{vmatrix} = 0$$

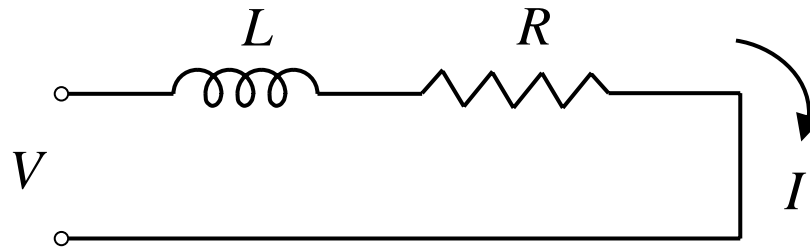
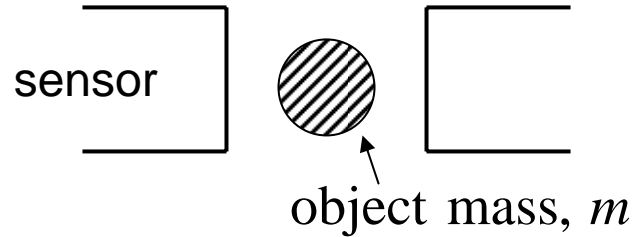
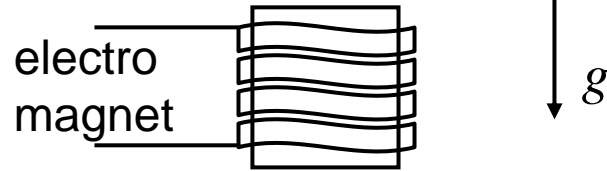
$$s^2 + \frac{g}{l} = 0$$

# PROJECT

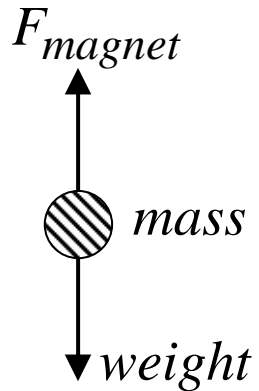
## PENDULUM



# MAGLEV



**Magnet**



$$F_{magnet} = f(k, I, x)$$

- Derive system dynamic equations
- Linearize, if needed
- Derive state space representation
- Find transfer function
- Find characteristic equation

# LAPLACE TRANSFORMS

Final Value Theorem (FVT)

$$f(\infty) \leftarrow \lim_{s \rightarrow 0} s F(s)$$

Condition: All the poles of  $F(s)$  other than  $s = 0$  must have negative non - zero real parts.

Poles: Are points at which the  $f(t)$  or its derivatives  $\rightarrow$  infinity

Zeros: Are points at which the  $f(t)$  equals zero

$$G(s) = \frac{(s + 2)(s + 10)}{s(s + 1)(s + 7)(s + 17)^2}$$

2 zeros, 5 poles

