the amplitude of the input to the saturating element is small. (This can be done by means of minor feedback-loop compensation.)

9-4 LAG-LEAD COMPENSATION

We shall first examine the frequency-response characteristics of the lag–lead compensator. Then we present the lag–lead compensation technique based on the frequency-response approach.

Characteristic of lag–lead compensator. Consider the lag–lead compensator given by

$$G_c(s) = K_c \left(\frac{s + \frac{1}{T_1}}{s + \frac{\gamma}{T_1}} \right) \left(\frac{s + \frac{1}{T_2}}{s + \frac{1}{\beta T_2}} \right) \qquad (9\text{–}3)$$

where $\gamma > 1$ and $\beta > 1$. The term

$$\frac{s + \frac{1}{T_1}}{s + \frac{\gamma}{T_1}} = \frac{1}{\gamma} \left(\frac{T_1 s + 1}{\frac{T_1}{\gamma} s + 1} \right) \qquad (\gamma > 1)$$

produces the effect of the lead network, and the term

$$\frac{s + \frac{1}{T_2}}{s + \frac{1}{\beta T_2}} = \beta \left(\frac{T_2 s + 1}{\beta T_2 s + 1} \right) \qquad (\beta > 1)$$

produces the effect of the lag network.

In designing a lag–lead compensator, we frequently chose $\gamma = \beta$. (This is not necessary. We can, of course, choose $\gamma \neq \beta$.) In what follows, we shall consider the case where $\gamma = \beta$. The polar plot of the lag–lead compensator with $K_c = 1$ and $\gamma = \beta$ becomes as shown in Figure 9–21. It can be seen that, for $0 < \omega < \omega_1$, the compensator acts as a lag

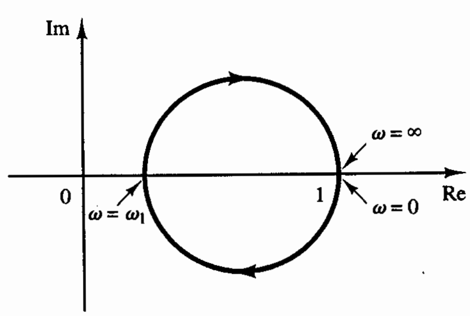

Figure 9–21
Polar plot of a lag–lead compensator given by Equation (9–3), with $K_c = 1$ and $\gamma = \beta$.

compensator, while for $\omega_1 < \omega < \infty$, it acts as a lead compensator. The frequency ω_1 is the frequency at which the phase angle is zero. It is given by

$$\omega_1 = \frac{1}{\sqrt{T_1 T_2}}$$

(To derive this equation, see Problem A–9–2.)

Figure 9–22 shows the Bode diagram of a lag–lead compensator when $K_c = 1$, $\gamma = \beta = 10$, and $T_2 = 10T_1$. Notice that the magnitude curve has the value 0 dB at the low- and high-frequency regions.

Lag–lead compensation based on the frequency-response approach. The design of a lag–lead compensator by the frequency-response approach is based on the combination of the design techniques discussed under lead compensation and lag compensation.

Let us assume that the lag–lead compensator is of the following form:

$$G_c(s) = K_c \frac{(T_1 s + 1)(T_2 s + 1)}{\left(\frac{T_1}{\beta}s + 1\right)(\beta T_2 s + 1)} = K_c \frac{\left(s + \frac{1}{T_1}\right)\left(s + \frac{1}{T_2}\right)}{\left(s + \frac{\beta}{T_1}\right)\left(s + \frac{1}{\beta T_2}\right)} \tag{9–4}$$

where $\beta > 1$. The phase lead portion of the lag–lead compensator (the portion involving T_1) alters the frequency-response curve by adding phase lead angle and increasing the phase margin at the gain crossover frequency. The phase lag portion (the portion involving T_2) provides attenuation near and above the gain crossover frequency and thereby allows an increase of gain at the low-frequency range to improve the steady-state performance.

We shall illustrate the details of the procedures for designing a lag–lead compensator by an example.

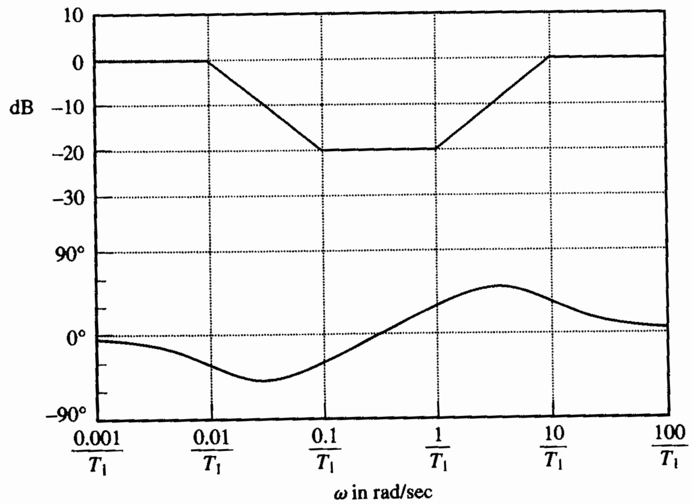

Figure 9–22 Bode diagram of a lag–lead compensator given by Equation (9–3) with $K_c = 1, \gamma = \beta = 10$, and $T_2 = 10T_1$.

EXAMPLE 9–3 Consider the unity-feedback system whose open-loop transfer function is

$$G(s) = \frac{K}{s(s+1)(s+2)}$$

It is desired that the static velocity error constant be 10 $\sec^{-1}$, the phase margin be 50°, and the gain margin be 10 dB or more.

Assume that we use the lag–lead compensator given by Equation (9–4). The open-loop transfer function of the compensated system is $G_c(s)G(s)$. Since the gain K of the plant is adjustable, let us assume that $K_c = 1$. Then, $\lim_{s\to 0} G_c(s) = 1$.

From the requirement on the static velocity error constant, we obtain

$$K_v = \lim_{s\to 0} sG_c(s)G(s) = \lim_{s\to 0} sG_c(s)\frac{K}{s(s+1)(s+2)} = \frac{K}{2} = 10$$

Hence,

$$K = 20$$

We shall next draw the Bode diagram of the uncompensated system with $K = 20$, as shown in Figure 9–23. The phase margin of the uncompensated system is found to be −32°, which indicates that the uncompensated system is unstable.

The next step in the design of a lag–lead compensator is to choose a new gain crossover frequency. From the phase angle curve for $G(j\omega)$, we notice that $\angle G(j\omega) = -180°$ at $\omega = 1.5$ rad/sec. It is convenient to choose the new gain crossover frequency to be 1.5 rad/sec so that the phase-lead angle required at $\omega = 1.5$ rad/sec is about 50°, which is quite possible by use of a single lag–lead network.

Once we choose the gain crossover frequency to be 1.5 rad/sec, we can determine the corner frequency of the phase lag portion of the lag–lead compensator. Let us choose the corner frequency $\omega = 1/T_2$ (which corresponds to the zero of the phase-lag portion of the compensator) to be 1 decade below the new gain crossover frequency, or at $\omega = 0.15$ rad/sec.

Recall that for the lead compensator the maximum phase lead angle ϕ_m is given by Equation (9–1), where α in Equation (9–1) is $1/\beta$ in the present case. By substituting $\alpha = 1/\beta$ in Equation (9–1), we have

$$\sin\phi_m = \frac{1 - \dfrac{1}{\beta}}{1 + \dfrac{1}{\beta}} = \frac{\beta - 1}{\beta + 1}$$

Notice that $\beta = 10$ corresponds to $\phi_m = 54.9°$. Since we need a 50° phase margin, we may choose $\beta = 10$. (Note that we will be using several degrees less than the maximum angle, 54.9°.) Thus,

$$\beta = 10$$

Then the corner frequency $\omega = 1/\beta T_2$ (which corresponds to the pole of the phase lag portion of the compensator) becomes $\omega = 0.015$ rad/sec. The transfer function of the phase lag portion of the lag–lead compensator then becomes

$$\frac{s + 0.15}{s + 0.015} = 10\left(\frac{6.67s + 1}{66.7s + 1}\right)$$

The phase lead portion can be determined as follows: Since the new gain crossover frequency is $\omega = 1.5$ rad/sec, from Figure 9–23, $G(j1.5)$ is found to be 13 dB. Hence, if the lag–lead compen-

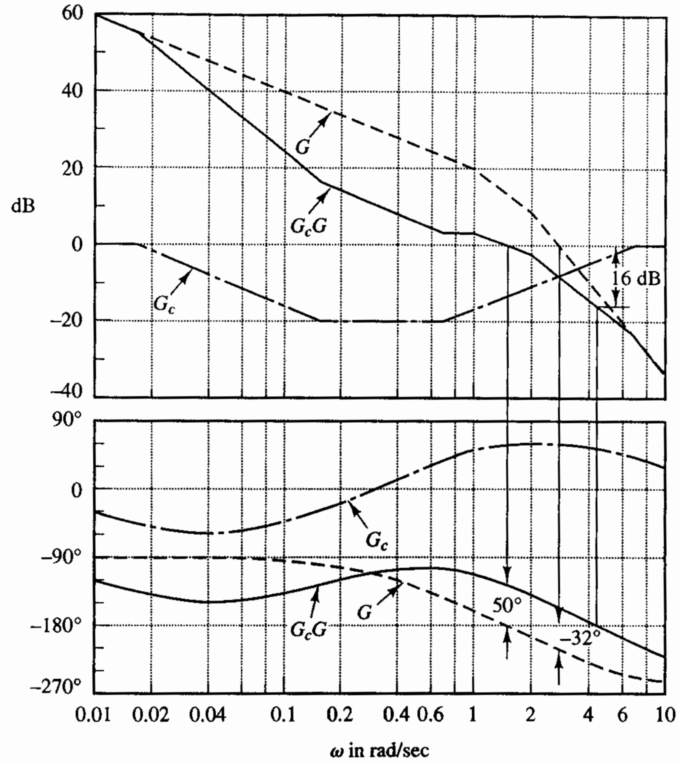

Figure 9–23 Bode diagrams for the uncompensated system, the compensator, and the compensated system. (G: uncompensated system, G_c: compensator, G_cG: compensated system.)

sator contributes -13 dB at $\omega = 1.5$ rad/sec, then the new gain crossover frequency is as desired. From this requirement, it is possible to draw a straight line of slope 20 dB/decade, passing through the point (-13 dB, 1.5 rad/sec). The intersections of this line and the 0-dB line and -20-dB line determine the corner frequencies. Thus, the corner frequencies for the lead portion are $\omega = 0.7$ rad/sec and $\omega = 7$ rad/sec. Thus, the transfer function of the lead portion of the lag–lead compensator becomes

$$\frac{s + 0.7}{s + 7} = \frac{1}{10}\left(\frac{1.43s + 1}{0.143s + 1}\right)$$

Combining the transfer functions of the lag and lead portions of the compensator, we obtain the transfer function of the lag–lead compensator. Since we chose $K_c = 1$, we have

$$G_c(s) = \left(\frac{s + 0.7}{s + 7}\right)\left(\frac{s + 0.15}{s + 0.015}\right) = \left(\frac{1.43s + 1}{0.143s + 1}\right)\left(\frac{6.67s + 1}{66.7s + 1}\right)$$

The magnitude and phase-angle curves of the lag–lead compensator just designed are shown in Figure 9–23. The open-loop transfer function of the compensated system is

$$G_c(s)G(s) = \frac{(s + 0.7)(s + 0.15)20}{(s + 7)(s + 0.015)s(s + 1)(s + 2)}$$

$$= \frac{10(1.43s + 1)(6.67s + 1)}{s(0.143s + 1)(66.7s + 1)(s + 1)(0.5s + 1)} \tag{9–5}$$

The magnitude and phase-angle curves of the system of Equation (9–5) are also shown in Figure 9–23. The phase margin of the compensated system is 50°, the gain margin is 16 dB, and the static velocity error constant is 10 sec^{-1}. All the requirements are therefore met, and the design has been completed.

Figure 9–24 shows the polar plots of the uncompensated system and compensated system. The $G_c(j\omega)G(j\omega)$ locus is tangent to the $M = 1.2$ circle at about $\omega = 2$ rad/sec. Clearly, this indicates that the compensated system has satisfactory relative stability. The bandwidth of the compensated system is slightly larger than 2 rad/sec.

In the following we shall examine the transient-response characteristics of the compensated system. (The uncompensated system is unstable.) The closed-loop transfer function of the compensated system is

$$\frac{C(s)}{R(s)} = \frac{95.381s^2 + 81s + 10}{4.7691s^5 + 47.7287s^4 + 110.3026s^3 + 163.724s^2 + 82s + 10}$$

The unit-step and unit-ramp response curves obtained with MATLAB are shown in Figures 9–25 and 9–26, respectively.

Note that the designed closed-loop control system has the following closed-loop zeros and poles:

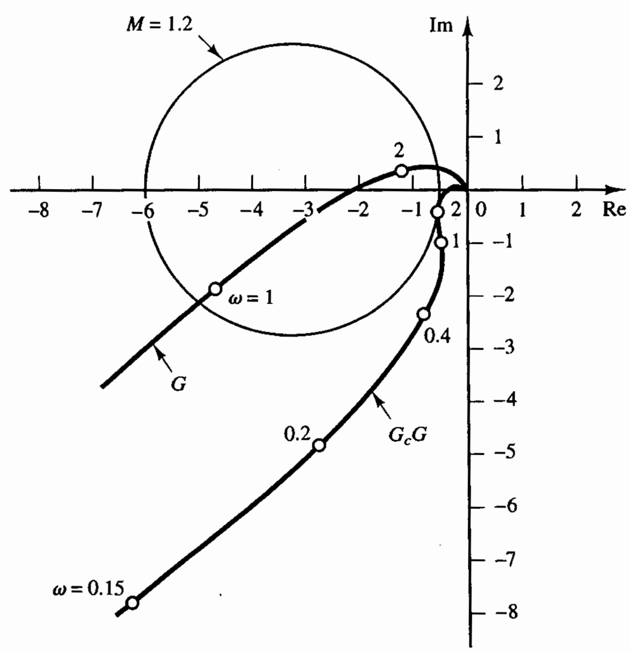

Figure 9–24
Polar plots of the uncompensated system and the compensated system. (G: uncompensated system, G_cG: compensated system.)

$$\text{Zeros at } s = -0.1499, \quad s = -0.6993$$

$$\text{Poles at } s = -0.8973 \pm j1.4439$$

$$s = -0.1785, \qquad s = -0.5425, \qquad s = -7.4923$$

The pole at $s = -0.1785$ and zero at $s = -0.1499$ are located very close to each other. Such a pair of pole and zero produces a long tail of small amplitude in the step response, as seen in Figure 9–25. Also, the pole at $s = -0.5425$ and zero at $s = -0.6993$ are located fairly close to each other. This pair adds an amplitude to the long tail.

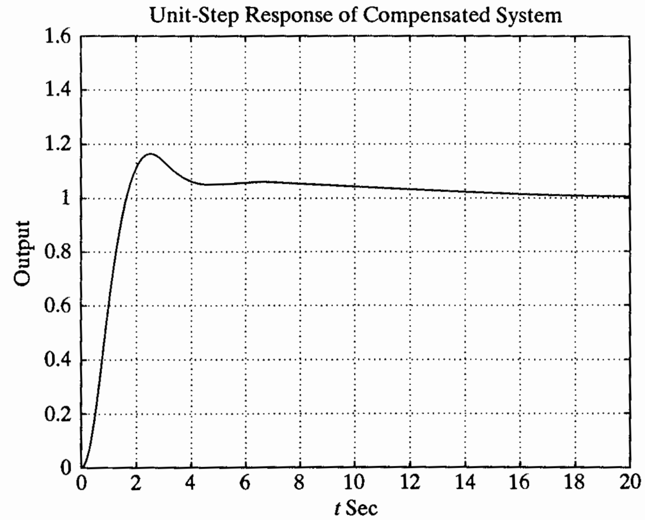

Figure 9–25 Unit-step response of the compensated system (Example 9–3).

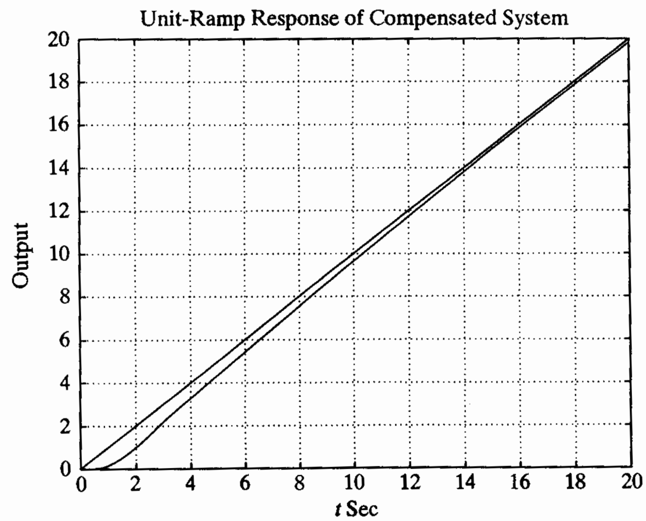

Figure 9–26 Unit-ramp response of the compensated system (Example 9–3).

9–5 CONCLUDING COMMENTS

This chapter has presented detailed procedures for designing lead, lag, and lag–lead compensators by the use of simple examples. We have shown that the design of a compensator to satisfy the given specifications (in terms of the phase margin and gain margin) can be carried out in the Bode diagram in a simple and straightforward manner. Note that a satisfactory design of a compensator for a complex system may require a creative application of these basic design principles.

Comparison of lead, lag and lag–lead compensation

1. Lead compensation achieves the desired result through the merits of its phase-lead contribution, whereas lag compensation accomplishes the result through the merits of its attenuation property at high frequencies. (In some design problems both lag compensation and lead compensation may satisfy the specifications.)

2. Lead compensation is commonly used for improving stability margins. Lead compensation yields a higher gain crossover frequency than is possible with lag compensation. The higher gain crossover frequency means larger bandwidth. A large bandwidth means reduction in the settling time. The bandwidth of a system with lead compensation is always greater than that with lag compensation. Therefore, if a large bandwidth or fast response is desired, lead compensation should be employed. If, however, noise signals are present, then a large bandwidth may not be desirable, since it makes the system more susceptible to noise signals because of increase in the high-frequency gain.

3. Lead compensation requires an additional increase in gain to offset the attenuation inherent in the lead network. This means that lead compensation will require a larger gain than that required by lag compensation. A larger gain, in most cases, implies larger space, greater weight, and higher cost.

4. Lag compensation reduces the system gain at higher frequencies without reducing the system gain at lower frequencies. Since the system bandwidth is reduced, the system has a slower speed to respond. Because of the reduced high-frequency gain, the total system gain can be increased, and thereby low-frequency gain can be increased and the steady-state accuracy can be improved. Also, any high-frequency noises involved in the system can be attenuated.

5. If both fast responses and good static accuracy are desired, a lag–lead compensator may be employed. By use of the lag–lead compensator, the low-frequency gain can be increased (which means an improvement in steady-state accuracy), while at the same time the system bandwidth and stability margins can be increased.

6. Although a large number of practical compensation tasks can be accomplished with lead, lag, or lag–lead compensators, for complicated systems, simple compensation by use of these compensators may not yield satisfactory results. Then, different compensators having different pole–zero configurations must be employed.

Graphical comparison. Figure 9–27(a) shows a unit-step response curve and unit-ramp response curve of an uncompensated system. Typical unit-step response and unit-ramp response curves for the compensated system using a lead, lag, and lag–lead network, respectively, are shown in Figures 9–27(b), (c), and (d). The system with a lead

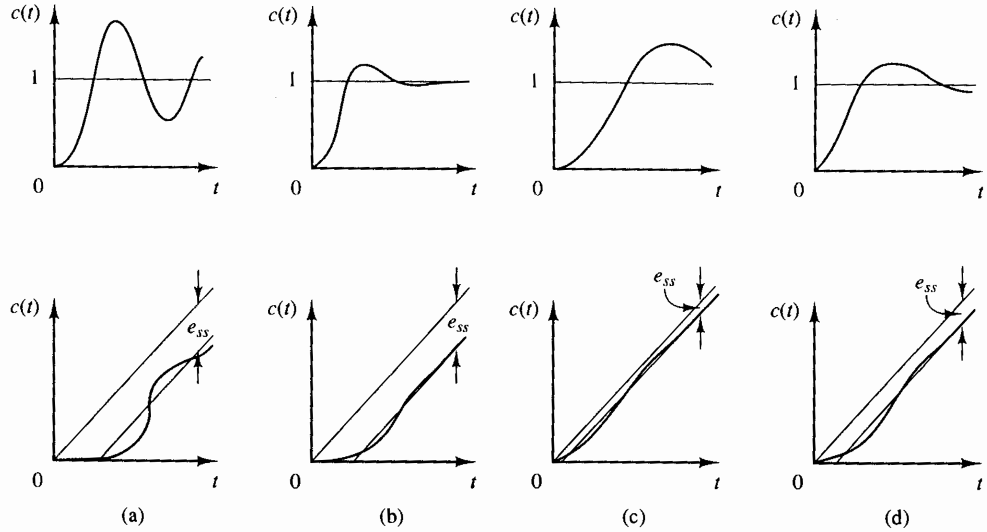

Figure 9–27 Unit-step response curves and unit-ramp response curves. (a) Uncompensated system; (b) lead compensated system; (c) lag compensated system; (d) lag–lead compensated system.

compensator exhibits the fastest response, while that with a lag compensator exhibits the slowest response, but with marked improvements in the unit-ramp response. The system with a lag–lead compensator will give a compromise; reasonable improvements in both the transient response and steady-state response can be expected. The response curves shown depict the nature of improvements that may be expected from using different types of compensators.

Feedback compensation. A tachometer is one of the rate feedback devices. Another common rate feedback device is the rate gyro. Rate gyros are commonly used in aircraft autopilot systems.

Velocity feedback using a tachometer is very commonly used in positional servo systems. It is noted that, if the system is subjected to noise signals, velocity feedback may generate some difficulty if a particular velocity feedback scheme performs differentiation of the output signal. (The result is the accentuation of the noise effects.)

Cancellation of undesirable poles. Since the transfer function of elements in cascade is the product of their individual transfer functions, it is possible to cancel some undesirable poles or zeros by placing a compensating element in cascade, with its poles and zeros being adjusted to cancel the undesirable poles or zeros of the original system. For example, a large time constant T_1 may be canceled by use of the lead network $(T_1s + 1)/(T_2s + 1)$ as follows:

$$\left(\frac{1}{T_1s + 1}\right)\left(\frac{T_1s + 1}{T_2s + 1}\right) = \frac{1}{T_2s + 1}$$

If T_2 is much smaller than T_1, we can effectively eliminate the large time constant T_1. Figure 9–28 shows the effect of canceling a large time constant in step transient response.